ISRAEL JOURNAL OF MATHEMATICS, Vol. 38, No. 3, 1981

AN INTEGRAL STRONG LAW OF LARGE NUMBERS FOR PROCESSES WITH INDEPENDENT INCREMENTS

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ABSTRACT

For a stochastically continuous stochastic process with independent increments over $D[0,T]$, let $N(t,\varepsilon)$ be the number of sample function jumps that occur in the interval $[0,t]$ of sizes less than $-\varepsilon$ or greater than ε, where $\varepsilon > 0$. Let $M(t,\varepsilon) = \mathrm{E}N(t,\varepsilon)$, and assume $M(t,0+) = \infty$ for $0 < t \leqq T$. If $\lim_{\varepsilon\downarrow 0}(M(t,\varepsilon)/M(T,\varepsilon))$ exists and is positive for each $t \in (0,T]$, then $\lim_{\varepsilon\downarrow 0}(N(t,\varepsilon)/M(T,\varepsilon)) = 1$ for all $t \in (0,T]$ with probability one.

An integral strong law of large numbers for stochastically continuous processes with independent increments is obtained here. In brief, consider the process as a measure over $D[0,T]$, and let $N(t,\varepsilon)$ denote the number of sample function jumps during $[0,t]$ for which the absolute values of their sizes are greater than $\varepsilon > 0$. Let $M(t,\varepsilon) = \mathrm{E}N(t,\varepsilon)$, and assume that $M(t,0+) = \infty$ for all $t \in (0,T]$. Then it will be shown that the probability that $\lim_{\varepsilon\downarrow 0}(N(t,\varepsilon)/M(t,\varepsilon)) = 1$ for all $t \in (0,T]$ is one if $\lim_{\varepsilon\downarrow 0}(M(t,\varepsilon)/M(T,\varepsilon))$ exists and is positive for each $t \in (0,T]$.

Let us first establish some notation and definitions needed here for stochastically continuous stochastic processes with independent increments. Let $\{X(t), 0 \leqq t \leqq T\}$ denote such a process defined over $D[0,T]$, the space of all real-valued functions over $[0,T]$ which are continuous from the right and have limits from the left at each $t \in [0,T]$, and assume $T < \infty$. At each $t \in [0,T]$, the distribution function of $X(t)$ is known to be infinitely divisible with centering function $\gamma(t)$, variance of Gaussian component $\sigma^2(t)$, and with Lévy spectral measure $M_t(\cdot)$. For a fixed Borel set A whose closure does not contain 0, $M_t(A)$

[†] The research of Howard G. Tucker was supported in part by the National Science Foundation, Grant No. MCS76–03591A01.

Received May 15, 1979

is finite and is continuous and non-decreasing in t; it is known to be the expectation of the number of jumps of a sample function that occur during $[0, t]$ and whose sizes are in A. Associated with such a process is a measure μ defined over the Borel subsets of the *time-jump space*,

$$\Theta = [0, T] \times (\mathbf{R}^1 \backslash \{0\})$$

by

$$\mu((s, t] \times (a, b]) = M_t((a, b]) - M_s((a, b]),$$

where $0 < a < b$ or $a < b < 0$. Also associated with the process is a family of random variables, $\{\nu(A): A$ is a Borel subset of Θ whose closure and the line $x = 0$ are disjoint$\}$, defined as follows: for $\omega \in D[0, T]$, $\nu(A)(\omega)$ denotes the number of "time-jumps" of the sample function ω that are in A, i.e., the number of points (s, x) in A which satisfy the equation $\omega(s) - \omega(s - 0) = x$. For each such A, $\nu(A)$ is known to have a Poisson distribution with $\mathrm{E}\nu(A) = \mu(A)$. In addition, if $\{A_\lambda\}$ is a family of disjoint Borel sets in Θ whose closures are disjoint from the line $x = 0$, then $\{\nu(A_\lambda)\}$ are independent random variables. Our purpose is to prove the following theorem.

THEOREM. *If* (i) $M(t, 0+) = \infty$ *for* $t > 0$, *and* (ii) $\lim_{\varepsilon \downarrow 0}(M(t, \varepsilon)/M(T, \varepsilon))$ *exists and is positive for all* $t \in (0, T]$, *then*

$$\mathrm{P}\left[\lim_{\varepsilon \downarrow 0} \frac{N(t, \varepsilon)}{M(t, \varepsilon)} = 1 \text{ for all } t \in (0, T]\right] = 1. \tag{1}$$

The second hypothesis always holds for processes with stationary independent increments, in which case the limit is the uniform distribution function over $[0, T]$. However, (ii) does not necessarily hold for all stochastically continuous processes with independent increments, and examples are easily constructed of such processes for which (ii) does not hold. It does hold for a wide class of such processes, the limit always being a distribution function over $[0, T]$. (It should be noted that a distribution function so obtained need not be continuous, and there exist examples of processes where such limiting distribution functions are not continuous.) We now prove the theorem.

We first show, whether or not (ii) holds, that

$$\mathrm{P}\left[\lim_{\varepsilon \downarrow 0} \frac{N(t, \varepsilon)}{M(t, \varepsilon)} = 1\right] = 1. \tag{2}$$

Accordingly, let $\infty = \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \cdots$ be such that $\varepsilon_n \downarrow 0$ and $M(t, \varepsilon_1) > 0$. For $n = 1, 2, \cdots$, let us define $A_n \subset \Theta$ by

$$A_n = \{(s, x) \in \Theta : 0 \leqq s \leqq t, \varepsilon_n \leqq x < \varepsilon_{n-1}\}.$$

Noting that $\{\nu(A_n)\}$ are independent random variables with Poisson distributions with expectations $\{\mu(A_n)\}$, we have

$$M(t, \varepsilon_n) = \operatorname{var} N(t, \varepsilon_n) = \sum_{i=1}^{n} \operatorname{var} \nu(A_i).$$

We shall now use a special case of a strong law of large numbers due to V. V. Petrov (which can be found in [1]) which states: If $\{X_n\}$ is a sequence of independent random variables with finite variances and such that $\Sigma \operatorname{var}(X_n) = \infty$, then $(S_n - ES_n)/\operatorname{Var}(S_n) \to 0$ a.s., where $S_n = X_1 + \cdots + X_n$. In our case, $N(t, \varepsilon_n) = S_n$ and $M(t, \varepsilon_n) = ES_n = \operatorname{Var} S_n$, and thus

$$\lim_{n\to\infty} \frac{N(t, \varepsilon_n)}{M(t, \varepsilon_n)} = 1 \quad \text{a.s.} \tag{3}$$

In order to prove (2) we must show that (3) holds over the same set of probability one for all sequences $\varepsilon_n \downarrow 0$. We do this in two special cases which we then combine to give the general case.

Case (i). Suppose that, for fixed t, $M(t, \varepsilon)$ as a function of $\varepsilon > 0$ is a step function with denumerably many jumps, each of size not less than one. Let $\{\varepsilon_n\}$ be the points at which these jumps occur. There is no positive limit point of $\{\varepsilon_n\}$, and we may assume $\varepsilon_n \downarrow 0$. Let A be the set of probability one over which (3) occurs, and let $\delta_n \downarrow 0$ be arbitrary. It is easy to see that the sequence $\{N(t, \delta_n)/M(t, \delta_n)\}$ is a subsequence of $\{N(t, \varepsilon_n)/M(t, \varepsilon_n)\}$ but possibly including a finite number of repetitions of any term. Thus the new sequence based on $\{\delta_n\}$ converges over A to the same limit.

Case (ii). Now suppose that for fixed $t \in (0, T]$ and as a function of $\varepsilon > 0$, $M(t, \varepsilon)$ has no discontinuities of size in $[1, \infty)$, and use the fact that $M(t, 0+) = \infty$. In this case we may select $\varepsilon'_n \downarrow 0$ so that $0 < M(t, \varepsilon'_{n+1}) - M(t, \varepsilon'_n) \leqq 1$ for $n = 0, 1, 2, \cdots$. Let A be an event of probability one over which $N(t, \varepsilon'_n)/M(t, \varepsilon'_n) \to 1$ as $n \to \infty$. Now let $\varepsilon_n \downarrow 0$ be arbitrarily selected (but with $\varepsilon_1 \leqq \varepsilon'_1$). For $n = 1, 2, \cdots$, define k_n such that $\varepsilon'_{k_n} \geqq \varepsilon_n > \varepsilon'_{k_n+1}$. Then, over Ω,

$$\frac{N(t, \varepsilon'_{k_n})}{M(t, \varepsilon'_{k_n})} \cdot \frac{M(t, \varepsilon'_{k_n})}{M(t, \varepsilon'_{k_n+1})} \leqq \frac{N(t, \varepsilon_n)}{M(t, \varepsilon_n)} \leqq \frac{N(t, \varepsilon'_{k_n+1})}{M(t, \varepsilon'_{k_n+1})} \cdot \frac{M(t, \varepsilon'_{k_n+1})}{M(t, \varepsilon'_{k_n})}.$$

By the manner in which $\{\varepsilon'_n\}$ and A were selected, it follows that $N(t, \varepsilon_n)/M(t, \varepsilon_n) \to 1$ as $n \to \infty$ over A.

Case (iii). In the general case, we need only consider an $M(T,\varepsilon)$ which, as a function of $\varepsilon>0$, has infinitely many jumps of size not less than one at some $\varepsilon'_n \downarrow 0$. We may write $M(t,\varepsilon)=M_1(t,\varepsilon)+M_2(t,\varepsilon)$, where $M_2(t,\varepsilon)$ is a step function having only those jumps of $M(t,\varepsilon)$ whose sizes are $\geqq 1$. Let $N_2(t,\varepsilon)$ denote the number of jumps of the process during $[0,t]$ whose sizes are in $\{\varepsilon'_n \vee \varepsilon, n=1,2,\cdots\}$, and let $N_1(t,\varepsilon)=N(t,\varepsilon)-N_2(t,\varepsilon)$. Then by cases (i) and (ii) above,

$$\lim_{\varepsilon\downarrow 0}\frac{N_i(t,\varepsilon)}{M_i(t,\varepsilon)}=1, \qquad i=1,2,$$

over an event A of probability one. Thus, (2) follows by noting that one can write

$$\frac{N(t,\varepsilon)}{M(t,\varepsilon)}=\frac{\dfrac{N_1(t,\varepsilon)}{M_1(t,\varepsilon)}M_1(t,\varepsilon)+\dfrac{N_2(t,\varepsilon)}{M_2(t,\varepsilon)}M_2(t,\varepsilon)}{M_1(t,\varepsilon)+M_2(t,\varepsilon)}.$$

We now prove (1). Let $F(t)$ denote the limit in hypothesis (ii), and let $\Omega=D[0,T]$ be our probability space. For each $t\in(0,T]$, denote

$$S_t=\left[\lim_{\varepsilon\downarrow 0}\frac{N(t,\varepsilon)}{N(T,\varepsilon)}=F(t)\right].$$

By (2) and hypothesis (ii), S_t is measurable, and $P(S_t)=1$. Let D be any countable, dense subset of $[0,T]$ that includes all points at which F is not continuous, and set $A=\bigcap\{S_t : t\in D\}$. Now let $t\in[0,T]\backslash D$. Then F is continuous at t, and so, given any $\eta>0$, there exists a $\delta>0$ such that if $|t-u|<\delta$ and $u\in[0,T]$, then $|F(t)-F(u)|<\eta$. Let $0\leqq u<t<v\leqq T$ be such that $t-u<\delta$, $v-t<\delta$, $u\in D$ and $v\in D$. Since

$$\frac{N(u,\varepsilon)}{N(T,\varepsilon)}\leqq\frac{N(t,\varepsilon)}{N(T,\varepsilon)}\leqq\frac{N(v,\varepsilon)}{N(T,\varepsilon)}$$

everywhere over Ω, it follows that

$$F(t)-\eta\leqq\overline{\underline{\lim}}\frac{N(t,\varepsilon)}{N(T,\varepsilon)}\leqq F(t)+\eta$$

for all $\omega\in A$, i.e., over A

$$\lim_{\varepsilon\downarrow 0}\frac{N(t,\varepsilon)}{N(T,\varepsilon)}=F(t).$$

Thus, over A, $\lim_{\varepsilon\downarrow 0}(N(t,\varepsilon)/M(t,\varepsilon))=1$ for all $t\in(0,T]$. Q.E.D.

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